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On de Haan's Uniform Convergence Theorem

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Abstract. In [*Univ. Beograd Publ. Elektrotehn. Fak. Ser. Math.* **15** (2004), 85–86], we proved a new inequality for the Lebesgue measure and gave some applications. Here, we present as it new application new short and simple proof of de Haan's uniform convergence theorem.

A measurable function $g : (0, +\infty) \to (0, +\infty)$ is *translational $\mathcal{O}$-regularly varying* if

$$\overline{\lim_{s\to\infty}} \frac{g(s+t)}{g(s)} < +\infty \tag{1}$$

for each $t \in \mathbf{R}$. For properties and applications of this class of mappings see Tasković [5].

Let λ be a Lebesgue measure on the set of real numbers $\mathbf{R}$. In [1] we present the following inequality, and as its applications short and simple proofs of two famous Steinhaus' results.

Proposition 1 (I. Aranđelović [1]). *Let A be a measurable set of a positive measure and $\{x_n\}$ a bounded sequence of real numbers. Then*

$$\lambda(A) \leq \lambda(\overline{\lim}(x_n + A)).$$

Now, as new application of Proposition 1, we present the following new short and simple proof of de Haan's uniform convergence theorem [4]. For applications of this result see [2],[3] or [4].

Proposition 2 (L. de Haan [4]). *Let $f, g : \mathbf{R} \to \mathbf{R}$ be a measurable functions such that $g(s) > 0$ for any s, g is translational $\mathcal{O}$-regularly varying function and*

$$\lim_{s\to\infty} \frac{f(t+s) - f(s)}{g(s)} < +\infty,$$

for all $t \in \mathbf{R}$. Then

$$\lim_{s\to\infty} \sup_{t\in[a,b]} \frac{f(t+s) - f(s)}{g(s)} < +\infty,$$

for any $a, b \in \mathbf{R}$ such that $a < b$.

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Proof. By Egoroff's theorem follows that for any $a, b \in \mathbf{R}$ such that $a < b$, there exists measurable set $A \subseteq [a, b]$ such that $\lambda(A) > 0$ and

$$\lim_{s\to\infty} \sup_{t\in A} \frac{f(t+s) - f(s)}{g(s)} < +\infty.$$

Assume now that convergence is not uniform on $[a, b]$. Then there exists $\{x_n\} \subseteq [a, b]$ and $\{y_n\} \subseteq \mathbf{R}$ such that $\lim y_n = \infty$ and

$$\lim \frac{f(x_n + y_n) - f(y_n)}{g(y_n)} = \infty.$$

By Proposition 1 follows that

$$\lambda(\overline{\lim}(A - x_n)) \geq \lambda(A) > 0,$$

which implies that there exists $t \in \mathbf{R}$ and subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\{t + x_{n_j}\} \subseteq A$. Then

$$\frac{|f(x_{n_j} + y_{n_j}) - f(y_{n_j})|}{g(y_{n_j})} \leq$$

$$\leq \frac{|f(x_{n_j} + t + y_{n_j} - t) - f(y_{n_j} - t)|}{g(y_{n_j} - t)} \cdot \frac{g(y_{n_j} - t)}{g(y_{n_j})} + \frac{|f(y_{n_j} - t) - f(y_{n_j})|}{g(y_{n_j})}.$$

Now

$$\lim \frac{|f(x_{n_j} + t + y_{n_j} - t) - f(y_{n_j} - t)|}{g(y_{n_j} - t)} < +\infty, \tag{2}$$

because $\{t + x_{n_j}\} \subseteq A$ and $\lim(y_{n_j} - t) = \infty$. From (1), (2) and

$$\lim \frac{f(y_{n_j} - t) - f(y_{n_j})}{g(y_{n_j})} < +\infty$$

follows

$$\overline{\lim} \frac{f(x_{n_j} + y_{n_j}) - f(y_{n_j})}{g(y_{n_j})} < +\infty,$$

which is a contradiction. □

References

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